Testing the Difference Between Means - Dependent Samples

OBJECTIVES

- How to perform a *t*-test to test the mean of the differences for a population of paired data

To perform a two-sample hypothesis test with dependent samples, you will first find the difference d for each data pair:

$d = x_1 - x_2$. Difference between entries for a data pair

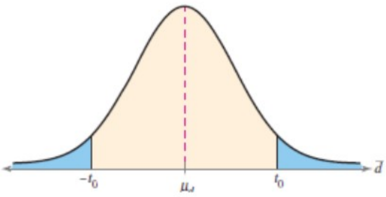
The test statistic is the mean $\overline{d}$ of these differences

$\overline{d} = \frac{\Sigma d}{n}$. Mean of the differences between paired data entries in the dependent samples

The following conditions are required to conduct the test.

1. The samples must be randomly selected.
2. The samples must be dependent (paired).
3. Both populations must be normally distributed.

If these requirements are met, then the **sampling distribution for $\overline{d}$, the mean of the differences of the paired data entries in the dependent samples,** is approximated by a t-distribution with $n - 1$ degrees of freedom, where n is the number of data pairs.



The standardized test statistic is

$$t = \frac{\overline{d} - \mu_d}{s_d/\sqrt{n}}$$

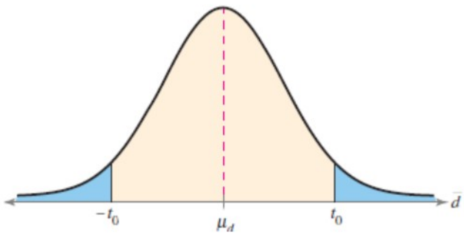
the standard deviation

$$s_d = \sqrt{\frac{\Sigma d^2 - \left[\frac{(\Sigma d)^2}{n}\right]}{n - 1}}$$

THE TWO-SAMPLE t-TEST FOR THE DIFFERENCE BETWEEN MEANS

The following symbols are used for the t-test for μ_d. Although formulas are given for the mean and standard deviation of differences, you should use a technology tool to calculate these statistics.

Symbol	Description
n	The number of pairs of data
d	The difference between entries for a data pair, $d = x_1 - x_2$
μ_d	The hypothesized mean of the differences of paired data in the population
$\overline{d}$	The mean of the differences between the paired data entries in the dependent samples $\overline{d} = \frac{\sum d}{n}$
s_d	The standard deviation of the differences between the paired data entries in the dependent samples $s_d = \sqrt{\frac{\sum (d - \overline{d})^2}{n - 1}}$



STUDY TIP

You can also calculate the standard deviation of the differences between paired data entries using the shortcut formula

$$s_d = \sqrt{\frac{\sum d^2 - \left[\frac{(\sum d)^2}{n}\right]}{n - 1}}.$$

THE TWO-SAMPLE t-TEST FOR THE DIFFERENCE BETWEEN MEANS

GUIDELINES

Using the t-Test for the Difference Between Means (Dependent Samples)

IN WORDS	IN SYMBOLS
1. State the claim mathematically and verbally. Identify the null and alternative hypotheses.	State H_0 and H_a.
2. Specify the level of significance.	Identify α.
3. Determine the degrees of freedom.	d.f. $= n - 1$
4. Determine the critical value(s).	Use Table 5 in Appendix B. If $n > 29$, use the last row (∞) in the t-distribution table.
5. Determine the rejection region(s).	
6. Calculate $\overline{d}$ and s_d.	$\overline{d} = \frac{\Sigma d}{n}$ $s_d = \sqrt{\frac{\Sigma (d - \overline{d})^2}{n - 1}}$
7. Find the standardized test statistic and sketch the sampling distribution.	$t = \frac{\overline{d} - \mu_d}{s_d/\sqrt{n}}$
8. Make a decision to reject or fail to reject the null hypothesis.	If t is in the rejection region, reject H_0. Otherwise, fail to reject H_0.
9. Interpret the decision in the context of the original claim.	

Testing the Difference Between Means - Dependent Samples

EXAMPLE 1 The *t*-Test for the Difference Between Means

A shoe manufacturer claims that athletes can increase their vertical jump heights using the manufacturer's new Strength Shoes®. The vertical jump heights of eight randomly selected athletes are measured. After the athletes have used the Strength Shoes® for 8 months, their vertical jump heights are measured again. The vertical jump heights (in inches) for each athlete are shown in the table. At $\alpha = 0.10$, is there enough evidence to support the manufacturer's claim? Assume the vertical jump heights are normally distributed. *(Adapted from Coaches Sports Publishing)*

Athlete	1	2	3	4	5	6	7	8
Vertical jump height (before using shoes)	24	22	25	28	35	32	30	27
Vertical jump height (after using shoes)	26	25	25	29	33	34	35	30

Solution

The claim is that "athletes can increase their vertical jump heights." In other words, the manufacturer claims that an athlete's vertical jump height before using the Strength Shoes® will be less than the athlete's vertical jump height after using the Strength Shoes®. Each difference is given by

$$d = (\text{jump height before shoes}) - (\text{jump height after shoes}).$$

The null and alternative hypotheses are

H_0: $\mu_d \geq 0$ and H_a: $\mu_d < 0$. (Claim)

Because the test is a left-tailed test, $\alpha = 0.10$, and d.f. $= 8 - 1 = 7$, the critical value is $t_0 = -1.415$. The rejection region is $t < -1.415$. Using the table at the left, you can calculate $\overline{d}$ and s_d as follows. Notice that the shortcut formula is used to calculate the standard deviation.

$$\overline{d} = \frac{\Sigma d}{n} = \frac{-14}{8} = -1.75$$

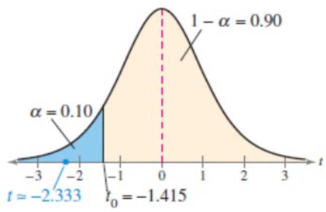
$$s_d = \sqrt{\frac{\Sigma d^2 - \left[\frac{(\Sigma d)^2}{n}\right]}{n-1}} = \sqrt{\frac{56 - \frac{(-14)^2}{8}}{8-1}} \approx 2.1213$$

The standardized test statistic is

$$t = \frac{\overline{d} - \mu_d}{s_d/\sqrt{n}} \quad \text{Use the } t\text{-test.}$$

$$\approx \frac{-1.75 - 0}{2.1213/\sqrt{8}} \quad \text{Assume } \mu_d = 0.$$

$$\approx -2.333.$$



The graph at the right shows the location of the rejection region and the standardized test statistic t. Because t is in the rejection region, you should decide to reject the null hypothesis.

Interpretation There is enough evidence at the 10% level of significance to support the shoe manufacturer's claim that athletes can increase their vertical jump heights using the new Strength Shoes®.

Testing the Difference Between Means - Dependent Samples

▸ Try It Yourself 1

Before	After
4.85	4.78
4.90	4.90
5.08	5.05
4.72	4.65
4.62	4.64
4.54	4.50
5.25	5.24
5.18	5.27
4.81	4.75
4.57	4.43
4.63	4.61
4.77	4.82

A shoe manufacturer claims that athletes can decrease their times in the 40-yard dash using the manufacturer's new Strength Shoes®. The 40-yard dash times of 12 randomly selected athletes are measured. After the athletes have used the Strength Shoes® for 8 months, their 40-yard dash times are measured again. The times (in seconds) are listed at the left. At $\alpha = 0.05$, is there enough evidence to support the manufacturer's claim? Assume the times are normally distributed. *(Adapted from Coaches Sports Publishing)*

a. Identify the *claim* and state H_0 and H_a.
b. Identify the *level of significance* α and the *degrees of freedom*.
c. Find the *critical value* t_0 and identify the *rejection region*.
d. *Calculate* $\overline{d}$ and s_d.
e. Use the t-test to find the *standardized test statistic t*. *Sketch* a graph.
f. *Decide* whether to reject the null hypothesis.
g. *Interpret* the decision in the context of the original claim.

STUDY TIP

If you prefer to use a technology tool for this type of test, enter the data in two columns and form a third column in which you calculate the difference for each pair. You can now perform a one-sample t-test on the difference column, as shown in Chapter 7.

Testing the Difference Between Means - Dependent Samples

EXAMPLE 2 The *t*-Test for the Difference Between Means

A state legislator wants to determine whether her performance rating (0–100) has changed from last year to this year. The following table shows the legislator's performance ratings from the same 16 randomly selected voters for last year and this year. At $\alpha = 0.01$, is there enough evidence to conclude that the legislator's performance rating has changed? Assume the performance ratings are normally distributed.

Voter	1	2	3	4	5	6	7	8
Rating (last year)	60	54	78	84	91	25	50	65
Rating (this year)	56	48	70	60	85	40	40	55

Voter	9	10	11	12	13	14	15	16
Rating (last year)	68	81	75	45	62	79	58	63
Rating (this year)	80	75	78	50	50	85	53	60

▸ Solution

If there is a change in the legislator's rating, there will be a difference between "this year's" ratings and "last year's" ratings. Because the legislator wants to see if there is a difference, the null and alternative hypotheses are

$H_0: \mu_d = 0$ and $H_a: \mu_d \neq 0$. (Claim)

Because the test is a two-tailed test, $\alpha = 0.01$, and d.f. $= 16 - 1 = 15$, the critical values are $-t_0 = -2.947$ and $t_0 = 2.947$. The rejection regions are $t < -2.947$ and $t > 2.947$.

Before	After	d	d^2
60	56	4	16
54	48	6	36
78	70	8	64
84	60	24	576
91	85	6	36
25	40	−15	225
50	40	10	100
65	55	10	100
68	80	−12	144
81	75	6	36
75	78	−3	9
45	50	−5	25
62	50	12	144
79	85	−6	36
58	53	5	25
63	60	3	9
		$\Sigma = 53$	$\Sigma = 1581$

Using the table at the left, you can calculate $\overline{d}$ and s_d as shown below.

$$\overline{d} = \frac{\Sigma d}{n} = \frac{53}{16} = 3.3125$$

$$s_d = \sqrt{\frac{\Sigma d^2 - \left[\frac{(\Sigma d)^2}{n}\right]}{n-1}}$$

$$= \sqrt{\frac{1581 - \frac{53^2}{16}}{16-1}} \approx 9.6797$$

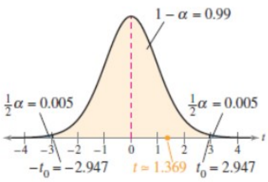
The standardized test statistic is

$$t = \frac{\overline{d} - \mu_d}{s_d/\sqrt{n}} \quad \text{Use the } t\text{-test.}$$

$$\approx \frac{3.3125 - 0}{9.6797/\sqrt{16}} \quad \text{Assume } \mu_d = 0.$$

$$\approx 1.369.$$

The graph at the right shows the location of the rejection region and the standardized test statistic t. Because t is not in the rejection region, you should fail to reject the null hypothesis.



Interpretation There is not enough evidence at the 1% level of significance to conclude that the legislator's performance rating has changed.

Testing the Difference Between Means - Dependent Samples

Try It Yourself 2

A medical researcher wants to determine whether a drug changes the body's temperature. Seven test subjects are randomly selected, and the body temperature (in degrees Fahrenheit) of each is measured. The subjects are then given the drug and, after 20 minutes, the body temperature of each is measured again. The results are listed below. At $\alpha = 0.05$, is there enough evidence to conclude that the drug changes the body's temperature? Assume the body temperatures are normally distributed.

Subject	1	2	3	4	5	6	7
Initial temperature	101.8	98.5	98.1	99.4	98.9	100.2	97.9
Second temperature	99.2	98.4	98.2	99	98.6	99.7	97.8

a. Identify the *claim* and state H_0 and H_a.
b. Identify the *level of significance* α and the *degrees of freedom*.
c. Find the *critical values* and identify the *rejection regions*.
d. *Calculate* $\overline{d}$ and s_d.
e. Use the t-test to find the *standardized test statistic* t. *Sketch* a graph.
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g. *Interpret* the decision in the context of the original claim.

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CLASSWORK

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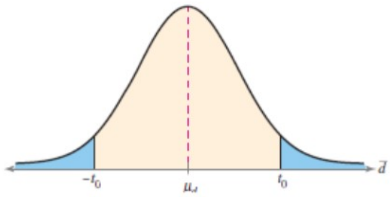

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